Physics Letters B 705 (2011) 157–160

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Birefringence and noncommutative structure of space–time

Marco Maceda *, Alfredo Macías

Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, A.P. 55-534, C.P. 09340, México D.F., Mexico

ARTICLE INFO

Article history:
Received 19 September 2011
Accepted 29 September 2011
Available online 5 October 2011
Editor: M. Cvetič

Keywords:
Birefringence
pp-waves
Noncommutative geometry

ABSTRACT

We analyze the phenomenon of birefringence of the electromagnetic field in the context of noncommutative geometry, using as background a deformed *pp*-wave solution to noncommutative Einstein's equations. The light-cone structure is determined using a generalized Fresnel equation characterizing the propagation of light in premetric vacuum electrodynamics.

1. Introduction

The theory of general relativity has been extended to different physical contexts, such as higher derivative theories, braneworlds and bi-metric theories, with the purpose of obtaining helpful insights, among others, into a quantum theory of gravity (see for example [1–3] and references therein). In [4], using the Groenewold–Moyal–Weyl $\star$-product, a 4-dimensional theory of deformed gravity due to a noncommutative structure has been proposed by Aschieri et al.

The first example involving a noncommutative structure of space–time was given by H. Snyder, who proposed a model [5,6] where coordinates in Minkowski space–time not longer commuted and had a discrete spectrum of values. Later on, in the 1990's, several works appeared dealing with the unification of gravity and the standard model using ideas of noncommutative geometry. It is expected that the existence of a noncommutative structure of space–time should manifest itself in high-energy processes, as for example in the Large Hadron Collider, where bounds on the noncommutative parameter of the deformed standard model may be obtained [7,8].

In this work we would like to discuss the effect of birefringence in electromagnetic waves as they move in a gravitational plane wave background [9], which is compatible with a noncommutative structure of space–time. For this, we take the construction in [4] as a specific model of a deformed theory of gravity. It should be mentioned that other 4-dimensional gravitational actions based on the $\star$-product have also been put forward [10,11], one of them [12] making use of the Seiberg–Witten map and having interesting features, e.g. the property that the star product is invariant under diffeomorphism transformations.

This work is organized as follows: in Section 2 we review the phenomenon of birefringence in general relativity. In Section 3 we present the solution describing noncommutative *pp*-waves, up to second order on the deformation parameter. Birefringence and light-cone structure are then discussed in Section 4. Finally, in Section 5, we comment on these results.

2. Birefringence in general relativity

It is well known [13] that Maxwell's equations in a four-dimensional curved space–time read:

$$f^{\alpha\beta}{}_{;\beta} = 4\pi j^{\alpha}, \qquad f_{[\alpha\beta;\gamma]} = 0, \tag{1}$$

where $f_{\alpha\beta}$ is the electromagnetic tensor; they can be written in equivalent Faraday form as the usual set of three-dimensional equations in flat space–time in terms of the electric field E_a, the magnetic field B_a, the electric displacement D_a, and the magnetization H_a,

$$\begin{aligned} &-D_{a,0} + \epsilon_{abc} H_{c,b} = 4\pi i_a, \qquad D_{a,a} = 4\pi\rho, \\ &B_{a,0} + \epsilon_{abc} E_{c,b} = 0, \qquad B_{a,a} = 0, \end{aligned} \tag{2}$$

using the following definitions

$$\begin{aligned} &E_a := f_{a0}, \qquad B_a := \frac{1}{2}\epsilon_{abc} f_{bc}, \\ &D_a := (-g)^{1/2} f^{0a}, \qquad H_a := \epsilon_{abc}(-g)^{1/2} f^{bc}, \\ &\rho := (-g)^{1/2} j^0, \qquad i_a := (-g)^{1/2} j^a, \end{aligned} \tag{3}$$

* Corresponding author.
E-mail addresses: mmac@xanum.uam.mx (M. Maceda), amac@xanum.uam.mx (A. Macías).

0370-2693
doi:10.1016/j.physletb.2011.09.115

together with the constitutive relations

$$D_a = \epsilon_{ab} E_b + \epsilon_{abc} g_b H_c,$$
$$B_a = -\epsilon_{abc} g_b E_c + \epsilon_{ab} H_b, \tag{4}$$

where

$$\epsilon_{ab} = \mu_{ab} := -\frac{(-g)^{1/2}}{g_{00}} g^{ab}, \qquad g_a := \frac{g_{a0}}{g_{00}}. \tag{5}$$

In the above expressions ϵ_{abc} is the standard three-dimensional Levi-Civita symbol with $\epsilon_{123} = +1$, g is the determinant of the metric, ρ is the charge density and i_a is the current density.

In Minkowski space–time, and for quiescent media, the three-dimensional tensors ϵ_{ab}, μ_{ab} are interpreted as corresponding to dielectric permittivity and magnetic permeability tensors. It should be stressed however, that the analogy with macroscopic electrodynamics makes only sense if the coordinates used are Cartesian ones. For example, the study of electromagnetic perturbations in Robertson–Walker universe within this context can be conveniently described [14] using the isotropic form of the metric.

In the absence of sources, Eqs. (2) become

$$D_{a,0} = \epsilon_{abc} H_{c,b}, \qquad B_{a,0} = -\epsilon_{abc} E_{c,b},$$
$$D_{a,a} = 0, \qquad B_{a,a} = 0. \tag{6}$$

If we look now for solutions of the form [15]

$$H_a, E_a, D_a, B_a \sim e^{i(k_a x^a - k_0 x^0)} = e^{ik_0(n_a x^a - x^0)}, \tag{7}$$

where $n_a := (g_{00})^{1/2} k_a/(e)^{1/2} k_0$, $e := \det(e_{ab})$, and e_{ab} is the vierbein, Maxwell's equations (6) become

$$-D_a = \epsilon_{abc} n_b H_c, \qquad B_a = \epsilon_{abc} n_b E_c,$$
$$n_a D_a = 0, \qquad n_a B_a = 0. \tag{8}$$

Inserting these expressions into the constitutive relations (4) and defining the tensor ρ_{ab} as the inverse tensor of ϵ_{ab}, i.e. $\rho_{ab}\epsilon_{bc} = \delta_{ac}$, we obtain [15]

$$\left[\epsilon_{as} - (n_b \epsilon_{bac} - \nu_{ac})\rho_{cd}(n_m \epsilon_{msd} - \nu_{sd})\right] E_s = 0, \tag{9}$$

where $\nu_{ac} := \epsilon_{abc} g_b$, $\nu_{ac} = -\nu_{ca}$. In order to have nontrivial solutions E_a, the determinant of the above coupled system of equations should vanish. This condition gives the Fresnel equation [16], which takes the form

$$\left[\det\left|\epsilon_{ab} + i(n_c \epsilon_{cab} - \nu_{ab})\right|\right]^2 \det|\rho_{ab}| = 0. \tag{10}$$

Hence, birefringence does not occur.

We would like to consider the influence of a noncommutative gravitational field on birefringence. Since we are mainly interested in the effects due to the background, we do not attempt here to deform Maxwell's electrodynamics. Such deformation would be of interest, for example, when studying the problem of self-energy of a charged particle.

3. Noncommutative Einstein's equations

The $\star$-product generalizes the standard point-wise multiplication of functions and involves an infinite number of terms, its definition being

$$(f \star g)(\boldsymbol{x}) := \exp\left[\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}\right] f(\boldsymbol{x}) g(\boldsymbol{y})|_{\boldsymbol{y}\to\boldsymbol{x}}, \tag{11}$$

with $\boldsymbol{x} := (x^\mu)$. The elements $\theta^{\mu\nu}$ are constant deformation parameters; it is clear that $\theta^{\mu\nu} = -\theta^{\nu\mu}$, and $[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$.

A noncommutative action for the gravitational field [4] has been proposed using the $\star$-product and appropriate generalizations of usual notions of general relativity. The noncommutative Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(E^\star \star R + c.c.\right), \tag{12}$$

where $E^\star$ means the $\star$-determinant defined as

$$E^\star := \frac{1}{4!}\epsilon^{\mu_1 \ldots \mu_4}{}_{A_1 \ldots A_4} E^{A_1}{}_{\mu_1} \star \cdots \star E^{A_4}{}_{\mu_4}, \tag{13}$$

with $\epsilon^{\mu_1 \ldots \mu_4}{}_{A_1 \ldots A_4}$ the Levi-Civita tensor, $R := G^{\mu\nu\star} \star R_{\mu\nu}$, $G_{\mu\nu} \star G^{\nu\rho\star} = \delta^\rho_\mu$, and

$$G_{\mu\nu} := \frac{1}{2}\left(E^A{}_\mu \star E^B{}_\nu + E^A{}_\nu \star E^B{}_\mu\right)\eta_{AB}. \tag{14}$$

η_{AB} is the usual Minkowski metric and the elements $E^A{}_\mu$ are real vector fields; they can be identified with the classical vierbein $e^A{}_\mu$ due to the condition

$$G_{\mu\nu}|_{\theta=0} = g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \eta_{AB}, \tag{15}$$

where $g_{\mu\nu}$ is the commutative metric.

The Christoffel symbols have expressions similar to the commutative case, namely

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\left(\partial^\star_\alpha \triangleright G_{\beta\gamma} + \partial^\star_\beta \triangleright G_{\alpha\gamma} - \partial^\star_\gamma \triangleright G_{\alpha\beta}\right) \star G^{\gamma\sigma\star}, \tag{16}$$

where $\partial^\star_\mu \triangleright := \partial_\mu$; they are symmetric, i.e. $\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}$. From them, the deformed Riemann tensor

$$R_{\mu\nu\rho}{}^\sigma := \partial^\star_\nu \triangleright \Gamma^\sigma_{\mu\rho} - \partial^\star_\mu \triangleright \Gamma^\sigma_{\nu\rho} + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \tag{17}$$

and the corresponding Ricci tensor $R_{\mu\nu} := R_{\mu\sigma\nu}{}^\sigma$ can be obtained. The deformed Einstein's equations in vacuum are simply $R_{\mu\nu} = 0$.

3.1. Deformed pp-wave solution

Consider the following line element [17]

$$ds^2 = 2\,du\,dv - 2h(u, x, y)(du)^2 - (dx)^2 - (dy)^2. \tag{18}$$

This family of space–times is generally known as plane fronted gravitational waves with parallel rays or *pp*-waves [18]. Space–time coordinates are denoted by $x^\mu = (u, v, x, y)$; u and v are standard retarded/advanced time coordinates.

In classical general relativity, the field equations in vacuum $R_{\mu\nu} = 0$ imply

$$\Delta h := h_{,xx} + h_{,yy} = 0, \tag{19}$$

i.e., h is a harmonic function of the variables x and y. In this expression a subscript denotes partial derivative, i.e., $X_{,x} := \partial_x X$, and $X_{,y} := \partial_y X$. Derivatives with respect to u and v are denoted as $X_{,u}$ and $X_{,v}$ respectively.

We consider the frame [19] $E^+ = \Lambda\,du$, $E^- = \Lambda^{-1}(dv - h\,du)$, $E^a = dx^a$, for the description of the noncommutative *pp*-waves in four dimensions. Here $x^a = (x, y)$, $h = h(u, x^a)$, $a = 2, 3$, and Λ is an arbitrary function; it is included in order to have a general scenario and at the same time, to keep the functional form of the noncommutative frame as close as possible with that of the commutative one.

We impose spatial noncommutativity $[x, y]_\star = i\theta$, by taking $\theta^{23} = -\theta^{32} := \theta$, as the only non-vanishing elements of the deformation parameter matrix $\theta^{\mu\nu}$. To simplify the calculations we choose to work on the linearized approximation $\Lambda = 1 + L$, and we assume also $h = h(u, x, y)$ and $L = L(u, x, y)$. Then we obtain

$$R_{uu} = R_{uu}^{(0)} + \theta^2 R_{uu}^{(2)} = -\Delta h + \frac{\theta^2}{2}\Delta s, \tag{20}$$

where

$$s := \frac{1}{4}(L_{,xx}h_{,yy} - 2L_{,xy}h_{,xy} + L_{,yy}h_{,xx}), \tag{21}$$

as the only non-vanishing component of the deformed Ricci tensor up to second order on θ. It follows that $R =$ up to this order as well.

Noncommutative *pp*-wave solution is then defined naturally through the conditions

$$R_{uu}^{(0)} = \Delta h = 0, \qquad R_{uu}^{(2)} = \Delta s = 0. \tag{22}$$

Given a harmonic function h, the function L is therefore determined from the second equation.

4. Noncommutative birefringence

Using Eq. (14) we find $G_{uu} = -2h + \theta^2 s, G_{uv} = G_{vu} = 1, G_{22} = G_{33} = -1$, as the only nontrivial deformed metric coefficients. Therefore, the corresponding noncommutative line element up to second order in θ is

$$ds^2 = 2\,du\,dv - \left(2h - \theta^2 s\right)(du)^2 - (dx)^2 - (dy)^2. \tag{23}$$

As a specific example, consider a plane wave solution with defining function $h = h_{ab}x^a x^b$, $a, b = 1, 2$, such that each element h_{ab} is constant. Then the equation $\Delta h = 0$ implies $h_{11} + h_{22} = 0$. The field equation $\Delta s = 0$ reduces to

$$(\Delta L)_{,xx}h_{22} + (\Delta L)_{,yy}h_{11} = 2(\Delta L)_{,xy}h_{12}, \tag{24}$$

or equivalently

$$(\Delta L)_{,xx} - (\Delta L)_{,yy} = \gamma(\Delta L)_{,xy}, \quad \gamma := 2\frac{h_{12}}{h_{22}}, \tag{25}$$

where we have used $h_{12} = h_{21}$, and assumed $h_{22} \neq 0$.

If $\gamma = 0$, then we see that

$$\Delta L = f_0(u, x + y) + f_1(u, x - y). \tag{26}$$

Here f_0 and f_1 are arbitrary functions of their arguments. The expression for s in Eq. (21) reduces in this case to

$$s = \frac{h_{11}}{4}(L_{,yy} - L_{,xx}). \tag{27}$$

To further proceed, we need to find an appropriate Cartesian coordinate system where Maxwell's equations in the curved background equation (23) may be interpreted as those in flat space together with constitutive relations. This can actually be achieved, using the results in [20]. The main requirement in this procedure is that the function

$$H := h - \frac{\theta^2}{2}s, \tag{28}$$

be a separable function. With our ansatz for h we have

$$H = h_{11}\left(x^2 - y^2\right) - \frac{\theta^2 h_{11}}{8}(L_{,yy} - L_{,xx}). \tag{29}$$

Since the first term is separable, we have then the condition

$$0 = (L_{,yy} - L_{,xx})_{,xy} = (L_{,xy})_{,yy} - (L_{,xy})_{,xx}, \tag{30}$$

from which we deduce

$$L_{,xy} = g_0(u, x + y) + g_1(u, x - y), \tag{31}$$

where g_0 and g_1 are arbitrary functions of their arguments. By defining now new variables $\alpha := x + y, \beta := x - y$, it can be shown that the most general expression for L, compatible with Eqs. (26) and (31), is given by

$$L(u, x, y) = \frac{1}{2}G_0^+(u, \alpha) + \frac{1}{2}G_1^-(u, \beta) + c_0\alpha\beta + d_0, \tag{32}$$

where $G_{0,\alpha\alpha}^+ = f_0(u, \alpha)$, $G_{1,\beta\beta}^- = f_1(u, \beta)$. Here $c_0 = c_0(u)$, and $d_0 = d_0(u)$, are arbitrary functions of their arguments. Using Eq. (32), we obtain thus for s the expression

$$s = \frac{h_{11}}{4}\left[-4c_0(u)\right] = -h_{11}c_0(u), \tag{33}$$

and therefore

$$H = h_{11}\left(x^2 - y^2\right) + \frac{\theta^2}{2}h_{11}c_0(u). \tag{34}$$

The diagonalization process can then be applied, leading to a metric of the form

$$ds^2 = 2\,du\,dv - F(u)^2\,dX^2 - G(u)^2\,dY^2, \tag{35}$$

where [20] $F_{,uu} = -f(u), G_{,uu} = f(u)$, and $f(u) := \frac{\partial^2 H}{\partial x^2} = 2h_{11} =$ const. The passage to a Cartesian coordinate system eliminates the dependence of the metric on the deformation parameter θ.

4.1. Light-cone structure

By using coordinates $T := \frac{1}{\sqrt{2}}(u + v), Z := \frac{1}{\sqrt{2}}(u - v)$, the line element Eq. (35) becomes

$$ds^2 = dT^2 - F(T + Z)^2\,dX^2 - G(T + Z)^2\,dY^2 - dZ^2. \tag{36}$$

In this Cartesian coordinate system, we have the following expressions $(a, b = 1, 2, 3)$

$$\epsilon_{ab} = \mu_{ab} = \mathrm{diag}(G/F, F/G, FG), \tag{37}$$

for the permittivity and permeability matrices. Furthermore, since $g_{0a} = 0$, there is no magnetic–electric cross terms and therefore the constitutive relations are simply $H_a = (\epsilon^{-1})_{ab}B_a$, and $D_a = \epsilon_{ab}E_b$.

To analyze the light-cone structure in this case, we need to consider the generalized Fresnel equation [21,22]

$$M_0k_0^4 + M_1k_0^3 + M_2k_0^2 + M_3k_0 + M_4 = 0, \tag{38}$$

where k_0 is the zeroth component of the 4-wave vector k^μ, while k_a are the spatial components, and

$$M_i := M^{a_1\ldots a_i}k_{a_1}\ldots k_{a_i}, \quad i = 1, \ldots, 4. \tag{39}$$

Explicit expressions for the elements $M^{a_1\ldots a_i}$ can be found in [23,21,22]; they are obtained from the 4th-order Tamm–Rubilar tensor density [23].

In our case, the Fresnel equation simplifies to

$$M_0k_0^4 + M_2k_0^2 + M_4 = 0, \tag{40}$$

where $M_0 = FG$, and

$$M_2 = -2\left(\frac{G}{F}k_1^2 + \frac{F}{G}k_2^2 + FGk_3^2\right),$$

$$M_4 = \frac{1}{FG}\left(\frac{G}{F}k_1^2 + \frac{F}{G}k_2^2 + FGk_3^2\right)^2. \tag{41}$$

Using these expressions, Eq. (40) can be written as

$$\left(k_0^2 - F^{-2}k_1^2 - G^{-2}k_1^2 - k_3^2\right)^2 = 0, \tag{42}$$

which implies that the quartic Fresnel wave surface reduces to a unique light-cone

$$g^{\mu\nu}k_\mu k_\nu = k_0^2 - F^{-2}k_1^2 - G^{-2}k_1^2 - k_3^2. \tag{43}$$

Hence, no birefringence appears in the noncommutative case.

5. Conclusions

We have shown that by going into a Cartesian coordinate system, we can eliminate the dependence on the deformation parameter θ of the noncommutative *pp*-wave metric for the choice $h = A_{11}(x^2 - y^2)$ of the classical part. This implies that the constitutive relations obtained from the line element Eq. (36) have no magnetic–electric cross terms. The resulting solution to the Fresnel equation shows a unique light-cone structure given by Eq. (43).

Therefore, no birefringence appears in the case of spatial noncommutativity. A reason for this is the ansatz used for the function h; its form was chosen having in mind the diagonalization process of the *pp*-wave metric at the classical level. This implies $h_{12} = 0$, which leads quite naturally to Eq. (34). We see then that the classical structure imposes strong constraints on its possible modifications.

A generalization to our ansatz is obtained by considering polynomial expressions of the form $h = \sum_{i=1}^{n} h_{a_1\ldots a_i} x^{a_1}\ldots x^{a_i}$, where $a_i = 1, 2$, and the coefficients $h_{a_1\ldots a_i}$ are chosen appropriately in order to satisfy the field equations and the separability condition needed for the transformation of the metric to a Cartesian coordinate system. Moreover, it would be also possible to include time–space noncommutativity. Work along these lines will be reported elsewhere.

Acknowledgements

This research was supported by CONACyT Grant No. 166041F3.

References

[1] T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82 (2010) 451.
[2] R. Maartens, K. Koyama, Living Rev. Rel. 13 (2010) 5.
[3] S. Deser, Bimetric gravity revisited, in: F.I. Cooperstock, L.P. Horowitz, J. Rosen (Eds.), Developments in General Relativity, Astrophysics and Quantum Theory: A Jubilee Volume in Honour of Nathan Rosen, IOP Publishing Co., Bristol, England, 1990, p. 77.
[4] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, J. Wess, Class. Quant. Grav. 22 (2005) 3511.
[5] H.S. Snyder, Phys. Rev. 71 (1947) 38.
[6] H.S. Snyder, Phys. Rev. 72 (1947) 68.
[7] A. Alboteanu, T. Ohl, R. Ruckl, Phys. Rev. D 76 (2007) 105018.
[8] M. Sakellariadou, Int. J. Mod. Phys. D 20 (2010) 785.
[9] A.B. Balakin, Class. Quant. Grav. 14 (1997) 2881.
[10] J.W. Moffat, Phys. Lett. B 491 (2000) 345.
[11] J.W. Moffat, Phys. Lett. B 493 (2000) 142.
[12] A.H. Chamseddine, Phys. Lett. B 504 (2001) 33.
[13] J. Plebanski, Phys. Rev. 118 (1959) 1396.
[14] B. Mashhoon, Phys. Rev. D 8 (1973) 4297.
[15] G.F. Glinskii, Radiophys. Quant. Electron. 21 (1978) 1041.
[16] L.D. Landau, E.M. Lifshitz, L.P. Pitaevskii, Electrodynamics of Continuous Media, 2nd ed., Pergamon Press, 1984.
[17] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, Exact Solutions of Einstein's Field Equations, 2nd ed., Cambridge University Press, 2009.
[18] A. Peres, Phys. Rev. Lett. 3 (1959) 571.
[19] M. Maceda, J. Madore, D.C. Robinson, Fuzzy-pp waves, in: B. Dragovich, B. Sazdovich (Eds.), I International Summer School in Modern Mathematical Physics, Belgrade Inst. of Physics Press, 2002, pp. 135–176.
[20] B.V. Ivanov, Diagonalization of pp-waves, arXiv:gr-qc/9705055, 1997.
[21] F.W. Hehl, Y.N. Obukhov, Found. Phys. 35 (2005) 2007.
[22] C. Lammerzahl, F.W. Hehl, Phys. Rev. D 70 (2004) 105022.
[23] G.F. Rubilar, Annalen Phys. 11 (2002) 717.